CLASSIFYING OPERATORS ON GENERALIZED DIFFERENTIAL AND DIFFERENCE RINGS

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1. Introduction

The theory of $\mathcal{D}$-rings was developed in a series of papers by Moosa and Scanlon (see [2], [3], [4]) as a unification and generalization of the theory of differential rings and that of difference rings. Given a algebra B of finite rank ℓ over a base ring A, a $\mathcal{D}$-ring is an A-algebra R together with an A-algebra homomorphism $e\colon R \to R \otimes_A B$. One may also consider the homomorphism e coordinate-wise given some basis ϵ for B by working with the sequence ∂ of operators on R such that $e(r) = \partial_0(r) \otimes \epsilon_0 + \cdots + \partial_{\ell-1}(r) \otimes \epsilon_{\ell-1}$. The latter presentation allows the theory of $\mathcal{D}$-rings to be axiomatized, while the former allows analysis using methods from commutative algebra. By imposing additional restrictions on the data, one can recover difference and differential rings, as outlined in Example 3.6 of [4]. Moosa and Scanlon focused on the model theory and geometry of $\mathcal{D}$-rings and their associated operators—particularly that of $\mathcal{D}$-fields of characteristic zero—but "[left] to to future work the systematical classification of the operators on A-algebras which it covers" ([4], Section 3.7). We shall consider this classification in the case when A is a field K.

We begin in Section 2 by presenting definitions and notation, then develop the algebra of $\mathcal{D}$-rings necessary for subsequent sections, culminating in a proof of the First Isomorphism Theorem. In Section 3, we give the construction of a free $\mathcal{D}$-ring, building on Example 3.8 of [4], and present a universal property which characterizes it. This example is used in the proof of an important proposition in Section 4. There, we turn to the classification of $\mathcal{D}$-operators, building on ideas introduced in [6]. The effect of the choice of basis ϵ is studied and a notion of equivalence under different bases for B is developed. We show that given certain assumptions on B, all $\mathcal{D}$-operators are equivalent to generalized derivations—that is, with a certain choice of basis, they satisfy a generalized Leibniz rule—and similarly that any operator satisfying this rule is a $\mathcal{D}$-operator. Finally, in Section 5, we consider the concept of competency—a certain condition conjectured in [1] to hold whenever K is algebraically closed and B is local, motivated by the classification of such algebras given in [5]. We show that to give a complete classification of $\mathcal{D}$-operators it suffices to consider the local case. We then prove a weaker condition which holds for all local K-algebras, whether or not K is algebraically closed.

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2. Preliminaries

All rings are commutative and unitary, and all algebras are associative. The ring unit is contained by all subrings and subalgebras and preserved by all ring and algebra homomorphisms. The natural numbers $\mathbb{N}$ are assumed to include 0.

Fix the following data for the duration of this paper:

(I) a base field K,
(II) a K-algebra B of finite rank ℓ,
(III) a basis $\epsilon = (\epsilon_0, \dots, \epsilon_{\ell-1})$ for B.

Observe that K plays the role of the ring A in [4]. Given a K-algebra R, we denote $R \otimes_K B$ by $\mathcal{D}(R)$. In particular, $\mathcal{D}(K) = K \otimes_K B \simeq B$, meaning that our notation is consistent with that of [4], which referred to the algebra B as $\mathcal{D}(K)$. We shall write $\mathcal{D}$ to denote the pair (B, ϵ).

Definition 2.1. A *$\mathcal{D}$-ring* is a K-algebra R together with an ℓ-tuple of operators $\partial = (\partial_0, \dots, \partial_{\ell-1})$ on R such that the map $e\colon R \to \mathcal{D}(R)$ given by

$$e(r) \coloneqq \sum_{i=0}^{\ell-1} \partial_i(r) \otimes \epsilon_i$$

is a K-algebra homomorphism. If R is an integral domain, we say that (R, ∂) is an *integral $\mathcal{D}$-ring*. The sequence ∂ is called a *$\mathcal{D}$-operator*. Observe that by insisting that $\partial_0 = \mathrm{id}_R$ we obtain the definition given in [4].

This section develops the foundations—morphisms, ideals, and quotients—of the algebra of $\mathcal{D}$-rings and $\mathcal{D}$-operators necessary for the rest of this paper. In particular, its content will be used in the construction of the free $\mathcal{D}$-ring in Section 3 and the partial classification of $\mathcal{D}$-operators in Section 4. Several examples of $\mathcal{D}$-rings and $\mathcal{D}$-operators and their associated homomorphisms can be found in Section 3 of [4].

Lemma 2.2. *If (R, ∂) is a $\mathcal{D}$-ring, then ∂_i is K-linear for $i = 0, \dots, \ell - 1$.*

Proof. For any $p, q \in R$ and $\lambda \in K$,

$$\sum_{i=0}^{\ell-1} \partial_i(p + \lambda q) \otimes \epsilon_i = e(p + \lambda q) = e(p) + \lambda e(q) = \sum_{i=0}^{\ell-1} (\partial_i(p) + \lambda \partial_i(q)) \otimes \epsilon_i.$$

The result follows by matching coefficients. □

Definition 2.3. Let (R, ∂) be a $\mathcal{D}$-ring. A *$\mathcal{D}$-subring* is a subalgebra S of R such that $\partial_i(S) \subseteq S$ for $i = 0, \dots, \ell - 1$. Note that (S, ∂^S) is also $\mathcal{D}$-ring, where ∂^S is the restriction of ∂ to S.

Definition 2.4. Let (R, ∂) and (R', ∂') be $\mathcal{D}$-rings. A *$\mathcal{D}$-morphism* from R to R' is a K-algebra homomorphism $\phi\colon R \to R'$ such that $\partial'_i \circ \phi = \phi \circ \partial_i$ for $i = 0, \dots, \ell - 1$. If ϕ is bijective, then we say that it is a *$\mathcal{D}$-isomorphism* and that (R, ∂) and (R', ∂') are *$\mathcal{D}$-isomorphic*.

Proposition 2.5. *Let (R, ∂) and (R', ∂') be $\mathcal{D}$-rings with associated homomorphisms e and e'. Suppose that $\phi\colon R \to R'$ is a homomorphism. Then the following are equivalent:*

(i) *ϕ is a $\mathcal{D}$-morphism,*
(ii) *$e' \circ \phi = (\phi \otimes \mathrm{id}_B) \circ e$.*

Proof.

$$\begin{aligned} \phi \text{ is a } \mathcal{D}\text{-morphism} &\iff \partial_i' \circ \phi = \phi \circ \partial_i \quad \text{for } i = 0, \dots, \ell-1 \\ &\iff \sum_{i=0}^{\ell-1} \partial_i'(\phi(r)) \otimes \epsilon_i = \sum_{i=0}^{\ell-1} \phi(\partial(r)) \otimes \epsilon_i \quad \text{for all } r \in R \\ &\iff e'(\phi(r)) = (\phi \otimes \mathrm{id}_B)(e(r)) \quad \text{for all } r \in R \\ &\iff e' \circ \phi = (\phi \otimes \mathrm{id}_B) \circ e. \end{aligned}$$

□

Remark 2.6. Proposition 2.5 shows that ϕ is a $\mathcal{D}$-morphism if and only if the following diagram commutes:

$$\begin{CD} R @>{e}>> \mathcal{D}(R) \\ @V{\phi}VV @VV{\phi\otimes \mathrm{id}_B}V \\ R' @>>{e'}> \mathcal{D}(R') \end{CD}$$

Definition 2.7. Let (R, ∂) be a $\mathcal{D}$-ring. A *$\mathcal{D}$-ideal* is an ideal $I \subseteq R$ such that $\partial_i(I) \subseteq I$ for $i = 0, \dots, \ell - 1$.

Proposition 2.8. *Let (R, ∂) be a $\mathcal{D}$-ring. Suppose that I is a $\mathcal{D}$-ideal of R. Then $(R/I, \bar{\partial})$ is a $\mathcal{D}$-ring, where $\bar{\partial}_i \colon R/I \to R/I$ is given by $\bar{\partial}_i(r + I) = \partial_i(r) + I$.*

Proof. We must show that $\bar{\partial}$ is well-defined. Suppose $p - q \in I$ for some $p, q \in R$. Then $\partial_i(p) - \partial_i(q) = \partial_i(p-q) \in I$ for $i = 0, \dots, \ell-1$. It follows that $\bar{\partial}$ is well-defined.

To show that $(R/I, \bar{\partial})$ is a $\mathcal{D}$-ring, we must show that the map $\bar{e} \colon R/I \to \mathcal{D}(R/I)$ given by

$$\bar{e}(r + I) = \sum_{i=0}^{\ell-1} \bar{\partial}_i(r + I) \otimes \epsilon_i$$

is a K-algebra homomorphism. Let $\phi \colon R \to R/I$ be the canonical quotient map. Since ϕ is surjective, it suffices to show that the following diagram commutes:

$$\begin{CD} R @>{e}>> \mathcal{D}(R) \\ @V{\phi}VV @VV{\phi\otimes \mathrm{id}_B}V \\ R/I @>>{\bar{e}}> \mathcal{D}(R/I) \end{CD}$$

For any $r \in R$,

$$\bar{e}(\phi(r)) = \bar{e}(r + I) = \sum_{i=0}^{\ell-1} \bar{\partial}(r + I) \otimes \epsilon_i = \sum_{i=0}^{\ell-1} (\partial(r) + I) \otimes \epsilon_i = (\phi \otimes \mathrm{id}_B)(e(r)).$$

Therefore the square commutes, implying that $(R/I, \bar{\partial})$ is a $\mathcal{D}$-ring. □

Theorem 2.9 (First Isomorphism Theorem for $\mathcal{D}$-rings)**.** *Let (R, ∂) and (R', ∂') be $\mathcal{D}$-rings. Suppose that $\phi \colon R \to R'$ is a $\mathcal{D}$-morphism. Then the following hold:*

(i) $\ker \phi$ *is a $\mathcal{D}$-ideal of R,*
(ii) $\phi(R)$ *is a $\mathcal{D}$-subring of R',*
(iii) $R/\ker \phi$ *and $\phi(R)$ are isomorphic as $\mathcal{D}$-rings when equipped with operators as in Proposition 2.8 and Definition 2.3 respectively.*

Proof.

(i) Since ϕ is a K-algebra homomorphism, $\ker\phi$ is an ideal of R. If $r \in \ker\phi$, then $\phi(\partial_i(r)) = \partial_i'(\phi(r)) = \partial_i(0) = 0$, so $\ker\phi$ is closed under ∂_i. Therefore $\ker\phi$ is a $\mathcal{D}$-ideal of R.

(ii) Similarly, $\phi(R)$ is a subring of R'. Since $\partial_i' \circ \phi = \phi \circ \partial_i$, $\phi(R)$ is closed under ∂_i'. Therefore $\phi(R)$ is a D-subring of R'.

(iii) Let $I = \ker\phi$. Let $S' = \phi(R)$. Define $\psi\colon R/I \to S'$ by $\psi(r+I) = \phi(r)$ for $i = 0, \ldots, \ell-1$. Then ψ is a well-defined K-algebra isomorphism. Define $\bar{\partial}\colon R/I \to R/I$ by $\bar{\partial}_i(r+I) = \partial_i(r) + I$. For ease of notation, denote the restriction of ∂' to S' simply by ∂'. For any $r \in R$,

$$\partial_i'(\psi(r+I)) = \partial_i'(\phi(r)) = \phi(\partial_i(r)) = \psi(\partial_i(r) + I) = \psi(\bar{\partial}_i(r+I)).$$

Therefore ψ is a $\mathcal{D}$-morphism. □

3. The Free $\mathcal{D}$-ring

Definition 3.1. We say that a $\mathcal{D}$-ring (R, ∂) is *generated by* an element $r \in R$ if there is no proper $\mathcal{D}$-subring of R containing r.

The following construction was given in Example 3.8 of [4] to illustrate the sense in which $\mathcal{D}$-operators are free—the $\mathcal{D}$-structure alone does not force any non-trivial functional equations to hold among the operators. Since it will be used in the proof of a result in Section 4, we repeat it here in more detail.

Definition 3.2. Define Σ to be the alphabet $\{\eth_0, \ldots, \eth_{\ell-1}\}$. Let $K\{x\}$ be the commutative polynomial algebra over K in variables ξx, where $\xi \in \Sigma^*$. Choose $a_{i,j,k}, c_i \in K$ such that

$$\epsilon_i \epsilon_j = \sum_{k=0}^{\ell-1} a_{i,j,k} \epsilon_k,$$

and

$$1_B = \sum_{k=0}^{\ell-1} c_k \epsilon_k.$$

Define K-linear operators $\eth = (\eth_0, \ldots, \eth_{\ell-1})$ on the K-vector space generators of $K\{x\}$ by

$$\begin{aligned} \eth_k(1) &= c_k, \\ \eth_k(\xi x) &= \eth_k \xi x \quad \text{for all } \xi \in \Sigma^*, \end{aligned}$$

and

$$\eth_k(\xi_1 x \cdots \xi_n x) = \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell-1} a_{i,j,k} \eth_i(\xi_1 x) \eth_j(\xi_2 x \cdots \xi_n x) \quad \text{for all } \xi_1, \ldots, \xi_n \in \Sigma^*,$$

where $n \geq 2$. A straightforward induction verifies that each $\eth_k$ is well defined and that the map $f\colon K\{x\} \to K\{x\} \otimes B$ given by

$$f(r) := \sum_{k=0}^{\ell-1} \eth_k(r) \otimes \epsilon_k$$

is a K-algebra homomorphism. Therefore $(K\{x\}, \eth)$ is a $\mathcal{D}$-ring, which we refer to as the *free $\mathcal{D}$-ring in one generator* and denote by $\mathrm{Free}(\mathcal{D})$.

Lemma 3.3. *The $\mathcal{D}$-ring* $\mathrm{Free}(\mathcal{D})$ *is generated by the indeterminate x.*

Proof. Suppose that $S \subseteq K\{x\}$ is a $\mathcal{D}$-subring containing x. We claim that $\xi x \in S$ for all $\xi \in \Sigma^*$. Proceed by induction on $\mathrm{length}(\xi)$. Observe that since $x \in S$, all strings ξx where $\mathrm{length}(\xi) = 0$ are in S.

Assume that $\xi x \in S$ for all $\xi \in \Sigma^*$ with $\mathrm{length}(\xi) = n$ for some $n \in \mathbb{N}$. Any string in Σ^* of length $n+1$ must be of the form $\eth_i \xi$ for some $i \in \{0, \dots, \ell - 1\}$ and $\xi \in \Sigma^*$, where $\mathrm{length}(\xi) = n$. Then $\eth_i \xi x = \eth_i(\xi x) \in S$ since S is closed under $\eth_i$. It follows that $\xi x \in S$ for all $\xi \in \Sigma^*$.

This shows that all the K-algebra generators of $K\{x\}$ are contained in S, and therefore $K\{x\} \subseteq S$. It follows that there is no proper $\mathcal{D}$-subring of $K\{x\}$ containing x, i.e., $\mathrm{Free}(\mathcal{D})$ is generated by x. ☐

Theorem 3.4 (Universal Property)**.** *Suppose that (R, ∂) is a $\mathcal{D}$-ring generated by a single element $r \in R$. Then there exists a surjective $\mathcal{D}$-morphism from* $\mathrm{Free}(\mathcal{D})$ *to (R, ∂) given by mapping the generator $x \in K\{x\}$ to r.*

Proof. Let $e\colon R \to \mathcal{D}(R)$ and $f\colon K\{x\} \to \mathcal{D}(K\{x\})$ be the K-algebra homomorphisms induced by the operators ∂ and $\eth$. Define $\phi\colon K\{x\} \to R$ by

(3.4.1) $$\phi(1) = 1,$$

(3.4.2) $$\phi(x) = r,$$

(3.4.3) $$\phi(\eth_i \xi x) = \partial_i(\phi(\xi x)) \quad \text{for } \xi \in \Sigma^* \text{ and } i \in \{0, \dots, \ell - 1\}.$$

Then ϕ gives a K-algebra homomorphism, since ϕ is defined on each K-algebra generator of $K\{x\}$. By Remark 2.6, to show that ϕ is a $\mathcal{D}$-morphism it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} K\{x\} & \xrightarrow{\;f\;} & \mathcal{D}(K\{x\}) \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi\otimes \mathrm{id}_B} \\ R & \xrightarrow[\;e\;]{} & \mathcal{D}(R) \end{array}$$

Since all maps in the diagram are K-algebra homomorphisms, it suffices to check that the diagram commutes for each K-algebra generator of $K\{x\}$. First note that

$$e(\phi(x)) = e(r) = \sum_{i=0}^{\ell-1} \partial_i(r) \otimes \epsilon_i = \sum_{i=0}^{\ell-1} \phi(\eth_i(x)) \otimes \epsilon_i = (\phi \otimes \mathrm{id}_B)(f(x)).$$

Let $j \in \{0, \dots, \ell - 1\}$ and $\xi \in \Sigma^*$. Then

$$\begin{aligned} e(\phi(\eth_j \xi x)) &= e(\partial_j(\phi(\xi x))) \\ &= \sum_{i=0}^{\ell-1} \partial_i(\partial_j(\phi(\xi x))) \otimes \epsilon_i \\ &= \sum_{i=0}^{\ell-1} \phi(\eth_i \eth_j \xi x)) \otimes \epsilon_i \\ &= (\phi \otimes \mathrm{id}_B)(f(\eth_j \xi x)). \end{aligned}$$

It follows that $e \circ \phi = (\phi \otimes \mathrm{id}_B) \circ f$ and therefore ϕ is a $\mathcal{D}$-morphism. By Theorem 2.9, $\phi(R)$ is a $\mathcal{D}$-subring of R containing r. Since R is generated by r, the image of ϕ is equal to R, i.e., ϕ is surjective. ☐

Remark 3.5. Note that in fact ϕ is completely determined by $\phi(x)$, as the identities (3.4.1) and (3.4.3) are necessary in order for ϕ to be a K-algebra homomorphism and a $\mathcal{D}$-morphism, respectively.

Corollary 3.6. *If (R, ∂) is a $\mathcal{D}$-ring generated by a single element, then R is $\mathcal{D}$-isomorphic to a quotient of $K\{x\}$.*

Proof. Define ϕ as in Theorem 3.4 so that $R \simeq K\{x\}/\ker\phi$. □

Corollary 3.7. *Suppose that (R, ∂) is a $\mathcal{D}$-ring generated by $r \in R$. Suppose that (R, ∂) satisfies the Universal Property given in Theorem 3.4. Then (R, ∂) is $\mathcal{D}$-isomorphic to* Free($\mathcal{D}$).

Proof. Let $\phi\colon K\{x\} \to R$ and $\psi\colon R \to K\{x\}$ be the surjective $\mathcal{D}$-morphisms given by defining $\phi(x) = r$ and $\psi(r) = x$. We claim that $\psi = \phi^{-1}$. It suffices to check that $\psi(\phi(\xi x)) = \xi x$ for all $\xi \in \Sigma^*$, as these are the K-algebra generators of $K\{x\}$.

Accordingly, fix some $\xi \in \Sigma^*$. Then $\xi = \eth_{t_0} \dots \eth_{t_k}$ for some natural numbers $t_0, \dots, t_k \leq \ell - 1$. It follows that

$$\psi(\phi(\xi x)) = \psi(\phi(\eth_{t_0} \dots \eth_{t_k} x)) = \psi((\partial_{t_0} \circ \dots \circ \partial_{t_k})(\phi(x))) = \eth_{t_0} \dots \eth_{t_k} \psi(\phi(x)) = \xi x$$

as both ϕ and ψ are $\mathcal{D}$-morphisms. □

4. A partial classification of $\mathcal{D}$-operators

Building on Sánchez and Moosa's preliminary work in [6], we give a classification of $\mathcal{D}$-operators given certain assumptions on B and ϵ. In particular, we develop a notion of equivalence of $\mathcal{D}$-operators which corresponds to a change of basis on B and show that when our assumptions hold, $\mathcal{D}$-operators on a ring R are precisely those which satisfy a generalized Leibniz rule.

Throughout this section, fix the following data:

(IV) a K-algebra B' of rank ℓ,
(V) a basis $\epsilon' = (\epsilon'_0, \dots, \epsilon'_{\ell-1})$ for B,

Additionally, fix a natural number m for the duration of this paper.

We shall write $\mathcal{D}'$ to denote the pair (B', ϵ'). We assume that $\mathbb{N}^m$ is equipped with the product order $\leq$. We shall denote the m-tuple $(0, \dots, 0) \in \mathbb{N}^m$ by 0 and the m-tuple of indeterminates $(x_1, \dots, x_m)$ by $\mathbf{x}$. For any $\alpha \in \mathbb{N}^m$, we write $\mathbf{x}^\alpha$ to denote the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

Definition 4.1. Let R be a K-algebra. Suppose that (R, ∂) is a $\mathcal{D}$-ring and (R, ∂') is a $\mathcal{D}'$-ring. We say that ∂ and ∂' are *equivalent* if there exists $M \in \mathrm{GL}_\ell(K)$ such that $\partial' = M\partial$, where ∂ and ∂' are considered as column vectors.

Proposition 4.2. *The following are equivalent:*

(i) *there exists $M \in \mathrm{GL}_\ell(K)$ such that if (R, ∂) is a $\mathcal{D}$-ring then $(R, M\partial)$ is a $\mathcal{D}'$-ring,*
(ii) *there exists $M \in \mathrm{GL}_\ell(K)$ such that if (R, ∂) is an integral $\mathcal{D}$-ring then $(R, M\partial)$ is an integral $\mathcal{D}'$-ring,*
(iii) *for all $\mathcal{D}$-rings (R, ∂), there exists $M \in \mathrm{GL}_\ell(K)$ such that $(R, M\partial)$ is a $\mathcal{D}'$-ring,*
(iv) *for all integral $\mathcal{D}$-rings (R, ∂), there exists $M \in \mathrm{GL}_\ell(K)$ such that $(R, M\partial)$ is an integral $\mathcal{D}'$-ring,*
(v) *there exists a K-algebra isomorphism $\phi\colon B \to B'$.*

Proof. (i) $\implies$ (ii), (i) $\implies$ (iii), and (ii) $\implies$ (iv) are immediate.

(iii) $\implies$ (v). For $i = 0, \ldots, \ell - 1$, define $\partial_i \colon B \to B$ by

$$\sum_{j=0}^{\ell-1} a_j \epsilon_j \mapsto a_i.$$

Then for any linear combination $r = \sum_{i=0}^{\ell-1} a_i \epsilon_i$ we have

$$\sum_{i=0}^{\ell-1} \partial_i(r) \otimes \epsilon_i = \sum_{i=0}^{\ell-1} a_i \otimes \epsilon_i = 1 \otimes \sum_{i=0}^{\ell-1} a_i \epsilon_i = 1 \otimes r.$$

It follows that the map $e \colon B \to \mathcal{D}(B)$ given by

$$e(r) = \sum_{i=0}^{\ell-1} \partial_i(r) \otimes \epsilon_i = 1 \otimes r$$

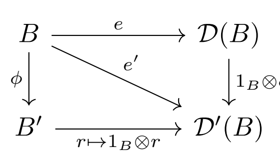
is a K-algebra homomorphism, i.e., (B, ∂) is a $\mathcal{D}$-ring. Let $M \in \mathrm{GL}_\ell(K)$ such that $(B, M\partial)$ is a $\mathcal{D}$'-ring. Let $\phi \colon B \to B'$ be the K-vector space isomorphism via K^ℓ given by M and the bases ϵ and ϵ'. That is, letting

$$M = \begin{bmatrix} \lambda_{0,0} & \cdots & \lambda_{0,\ell-1} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell-1,0} & \cdots & \lambda_{\ell-1,\ell-1} \end{bmatrix}$$

define ϕ by

$$\sum_{i=0}^{\ell-1} a_i \epsilon_i \mapsto \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell-1} \lambda_{i,j} a_j \epsilon'_i.$$

We will show that ϕ is in fact a K-algebra isomorphism. In particular, we show that the following diagram commutes, where $e' \colon B \to \mathcal{D}'(B)$ is the K-algebra homomorphism induced by $M\partial$.



For any linear combination $r = \sum_{i=0}^{\ell-1} a_i \epsilon_i$, we have

$$e'(r) = \sum_{i=0}^{\ell-1} (M\partial)_i(r) \otimes \epsilon'_i = \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell-1} \lambda_{i,j} \partial_j(r) \otimes \epsilon'_i = \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell-1} \lambda_{i,j} a_j \otimes \epsilon'_i = 1_B \otimes \phi(r).$$

It follows that ϕ is a K-algebra homomorphism. Since ϕ is a K-vector space isomorphism, its kernel is trivial, so ϕ is in fact a K-algebra isomorphism.

(iv) $\implies$ (v). For $i, j, k = 0, \ldots, \ell - 1$, fix $a_{i,j,k}, a'_{i,j,k}, c_k, c'_k \in K$ such that

$$1_B = \sum_{k=0}^{\ell-1} c_k \epsilon_k, \qquad 1_{B'} = \sum_{k=0}^{\ell-1} c'_k \epsilon'_k, \qquad \epsilon_i \epsilon_j = \sum_{k=0}^{\ell-1} a_{i,j,k} \epsilon_k, \qquad \epsilon'_i \epsilon'_j = \sum_{k=0}^{\ell-1} a'_{i,j,k} \epsilon'_k.$$

Consider the case when $(R, \partial) = \mathrm{Free}(\mathcal{D}) = (K\{x\}, \eth)$, as constructed in Example 3.2. Let $M \in \mathrm{GL}_\ell(K)$ such that $(K\{x\}, M\eth)$ is a $\mathcal{D}'$-ring. Write

$$M = \begin{bmatrix} \lambda_{0,0} & \cdots & \lambda_{0,\ell-1} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell-1,0} & \cdots & \lambda_{\ell-1,\ell-1} \end{bmatrix}$$

As before, let $\phi\colon B \to B'$ be the K-vector space isomorphism via K^ℓ given by M and the bases ϵ and ϵ'. We will show that ϕ is in fact a K-algebra isomorphism. Clearly ϕ is K-linear and bijective. Furthermore,

$$\begin{aligned} 1_R \otimes 1_{B'} &= e'(1_R) \\ &= \sum_{i=0}^{\ell-1} (M\eth)_i(1_R) \otimes \epsilon'_i \\ &= \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \lambda_{i,j}\eth_j(1_R) \otimes \epsilon'_i \\ &= \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \lambda_{i,j}c_j \otimes \epsilon'_i \\ &= 1_R \otimes \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \lambda_{i,j}c_j\epsilon'_i \\ &= 1_R \otimes \phi(c_0\epsilon_0 + \cdots + c_{\ell-1}\epsilon_{\ell-1}) \\ &= 1_R \otimes \phi(1_B). \end{aligned}$$

It follows that $\phi(1_B) = 1_{B'}$.

To show that ϕ is multiplicative, it suffices to show that $\phi(\epsilon_i\epsilon_j) = \phi(\epsilon_i)\phi(\epsilon_j)$ for $i, j = 0, \dots, \ell-1$. Note that

$$\phi(\epsilon_i\epsilon_j) = \sum_{k=0}^{\ell-1}\sum_{n=0}^{\ell-1} \lambda_{k,n}a_{i,j,n}\epsilon'_k \tag{4.2.1}$$

and

$$\phi(\epsilon_i)\phi(\epsilon_j) = \sum_{k=0}^{\ell-1}\sum_{n=0}^{\ell-1}\sum_{p=0}^{\ell-1} \lambda_{n,i}\lambda_{p,j}a'_{n,p,k}\epsilon'_k. \tag{4.2.2}$$

First note that the elements $\eth_i x \eth_j \eth_0 x$, where $i, j = 0, \ldots, \ell-1$, are distinct and linearly independent in $K\{x\}$. Furthermore,

$$\begin{aligned} e'(x\eth_0 x) &= \sum_{k=0}^{\ell-1} (M\partial)_k (x\eth_0 x) \otimes \epsilon'_k \\ &= \sum_{k=0}^{\ell-1}\sum_{n=0}^{\ell-1} \lambda_{k,n} \eth_n (x\eth_0 x) \otimes \epsilon'_k \\ &= \sum_{k=0}^{\ell-1}\sum_{n=0}^{\ell-1} \lambda_{k,n} \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} a_{i,j,n} \eth_i(x)\eth_j(\eth_0 x) \otimes \epsilon'_k \\ &= \sum_{k=0}^{\ell-1}\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \left(\sum_{n=0}^{\ell-1} \lambda_{k,n} a_{i,j,n}\right) \eth_i x \eth_j \eth_0 x \otimes \epsilon'_k \end{aligned}$$

and

$$\begin{aligned} e'(x)e'(\eth_0 x) &= \left(\sum_{n=0}^{\ell-1} \partial'_n(x) \otimes \epsilon'_n\right)\left(\sum_{p=0}^{\ell-1} \partial'_p(\eth_0 x) \otimes \epsilon'_p\right) \\ &= \left(\sum_{n=0}^{\ell-1}\sum_{i=0}^{\ell-1} \lambda_{n,i}\eth_i(x) \otimes \epsilon'_n\right)\left(\sum_{p=0}^{\ell-1}\sum_{j=0}^{\ell-1} \lambda_{p,j}\eth_j(\eth_0 x) \otimes \epsilon'_p\right) \\ &= \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1}\sum_{n=0}^{\ell-1}\sum_{p=0}^{\ell-1} \lambda_{n,i}\lambda_{p,j} \eth_i x \eth_j \eth_0 x \otimes \epsilon'_n \epsilon'_p \\ &= \sum_{k=0}^{\ell-1}\sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \left(\sum_{n=0}^{\ell-1}\sum_{p=0}^{\ell-1} \lambda_{n,i}\lambda_{p,j} a'_{n,p,k}\right) \eth_i x \eth_j \eth_0 x \otimes \epsilon'_k \end{aligned}$$

By linear independence, it follows that

$$\sum_{n=0}^{\ell-1} \lambda_{k,n} a_{i,j,n} = \sum_{n=0}^{\ell-1}\sum_{p=0}^{\ell-1} \lambda_{n,i}\lambda_{p,j} a'_{n,p,k}$$

for $i, j, k = 0, \ldots, \ell-1$. By (4.2.1) and (4.2.2), this shows that $\phi(\epsilon_i\epsilon_j) = \phi(\epsilon_i)\phi(\epsilon_j)$ for $i = 0, \ldots, \ell-1$. Therefore ϕ is a K-algebra isomorphism from B to B'.

(v) $\Longrightarrow$ (i). Let M be the matrix associated with the isomorphism ϕ with respect to the bases ϵ and ϵ', i.e.,

$$M = \begin{bmatrix} \lambda_{0,0} & \cdots & \lambda_{0,\ell-1} \\ \vdots & \ddots & \vdots \\ \lambda_{\ell-1,0} & \cdots & \lambda_{\ell-1,\ell-1} \end{bmatrix}$$

where

$$\phi(\epsilon_j) = \sum_{i=0}^{\ell-1} \lambda_{i,j} \epsilon'_i$$

for $j = 0, \dots, \ell - 1$. Let (R, ∂) be a $\mathcal{D}$-ring. Then the map $e\colon R \to \mathcal{D}(R)$ given by

$$e(r) = \sum_{i=0}^{\ell-1} \partial_i(r) \otimes \epsilon_i$$

is a K-algebra homomorphism. Define $e'\colon R \to \mathcal{D}'(R)$ by

$$e'(r) = \sum_{i=0}^{\ell-1} (M\partial)_i(r) \otimes \epsilon'_i.$$

We claim that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\;e\;} & \mathcal{D}(R) \\ & {\scriptstyle e'}\searrow & \downarrow{\scriptstyle \mathrm{id}_R \otimes \phi} \\ & & \mathcal{D}'(R) \end{array}$$

For any $r \in R$,

$$e'(r) = \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \lambda_{i,j}\partial_j(r) \otimes \epsilon'_i = \sum_{j=0}^{\ell-1} \partial_j(r) \otimes \sum_{i=0}^{\ell-1} \lambda_{i,j}\epsilon'_i = \sum_{j=0}^{\ell-1} \partial_j(r) \otimes \phi(\epsilon_j).$$

It follows that $e' = (\mathrm{id}_R \otimes \phi) \circ e$. Therefore e' is a homomorphism and $(R, M\partial)$ is a $\mathcal{D}'$-ring. □

Remark 4.3. We are particularly interested in the case when the $\mathcal{D}$-ring R is an integral domain, which is why the cases are treated separately in Proposition 4.2.

Definition 4.4. A nonempty finite set $N \subseteq \mathbb{N}^m$ is *Young-like* if whenever $\alpha \in N$ and $\beta < \alpha$ we have $\beta \in N$. Given a Young-like set N, we define N^* to be the set of minimal elements of $\mathbb{N}^m \setminus N$.

Definition 4.5. Let R be a K-algebra and N be a Young-like set. A *generalized derivation* of type N on R is a collection of operators $(\partial_\alpha)_{\alpha\in N}$ on R such that

(a) ∂_α is K-linear for all $\alpha \in N$,
(b) for all $p, q \in R$ and $\alpha \in N$,

$$\partial_\alpha(pq) = \sum_{\beta+\gamma=\alpha} \partial_\beta(p)\partial_\gamma(q),$$

(c) $\partial_0(1) = 1$.

Lemma 4.6. *Let R be a K-algebra and N be a Young-like set. Suppose that ∂ is a generalized derivation of type N on R. Then $\partial_\alpha(1) = 0$ for all $\alpha \in N \setminus \{0\}$.*

Proof. Proceed by induction on $|\alpha|$. If $|\alpha| = 1$, then

$$\partial_\alpha(1) = \sum_{\beta+\gamma=\alpha} \partial_\beta(1)\partial_\gamma(1) = \partial_0(1)\partial_\alpha(1) + \partial_\alpha(1)\partial_0(1) = 2\partial_\alpha(1),$$

so $\partial_\alpha(1) = 0$. Assume that $|\alpha| > 1$ and $\partial_\beta(1) = 0$ for all $\beta \in N$ with $1 \le |\beta| < |\alpha|$. Then

$$\partial_\alpha(1) = \sum_{\beta+\gamma=\alpha} \partial_\beta(1)\partial_\gamma(1) = 2\partial_0(1)\partial_\alpha(1) + \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\neq 0}} \partial_\beta(1)\partial_\gamma(1) = 2\partial_\alpha(1),$$

so $\partial_\alpha(1) = 0$. □

Definition 4.7. We say that an antichain $T \subseteq \mathbb{N}^m$ is *full* if the ideal I generated by the set $\{\mathbf{x}^\alpha : \alpha \in T\} \subseteq K[\mathbf{x}]$ is proper and zero-dimensional, i.e., the K-algebra $K[\mathbf{x}]/I$ is non-trivial and finite-dimensional as a K-vector space. Given a full antichain T, we define T^* to be the set $\{\alpha \in \mathbb{N}^m : \forall\, \beta \in T,\ \beta \not\leq \alpha\}$. Note that the only Young-like antichain of $\mathbb{N}^m$ is $\{0\}$, which is not full. This ensures that our notation is unambiguous. We say that $K[\mathbf{x}]/I$ is *of type* T.

Remark 4.8. Definition 4.7 seems at first somewhat unmotivated and arbitrary. However, the classification of local K-algebras of rank less than 7 given in [5] shows that

(i) all local K-algebras of rank less than 4 are isomorphic to an K-algebra of type T for some full antichain T,
(ii) of the 5 isomorphism classes of local K-algebras of rank 4, at least 4 admit a a representative of type T for some full antichain T,
(iii) of the 10 isomorphism classes of local K-algebras of rank 5, at least 7 admit a a representative of type T for some full antichain T,
(iv) of the 33 isomorphism classes of local K-algebras of rank 6, at least 14 admit a representative of type T for some full antichain T.

We shall discuss reducing the classification to the case when B is local in more detail in Section 5.

Proposition 4.9. *Suppose that $N \subseteq \mathbb{N}^m$ is a Young-like set and $T \subseteq \mathbb{N}^m$ is a full antichain. Then the following hold:*

(i) *N^* is a full antichain,*
(ii) *T^* is a Young-like set,*
(iii) *$(N^*)^* = N$,*
(iv) *$(T^*)^* = T$.*

Proof.

(i) Suppose that $\alpha, \beta \in N^*$ are distinct. If $\alpha < \beta$, then β is not minimal in $\mathbb{N}^m \setminus N$. Similarly we must have $\beta \not< \alpha$. It follows that α and β are incomparable. Therefore N^* is an antichain.

 It remains to show that N^* is full, i.e., that the ideal I generated by $\{\mathbf{x}^\alpha : \alpha \in N^*\} \subseteq K[\mathbf{x}]$ is proper and zero-dimensional. Since $0 \in N$, $0 \notin \mathbb{N}^*$, so $1 \notin I$ and therefore I is proper. We will show that all but finitely many monomials are in I, which shows that $K[\mathbf{x}]/I$ is finite-dimensional. Let $\alpha \in \mathbb{N}^m \setminus N$. By definition of N^*, there is some $\beta \in N^*$ such that $\beta \leq \alpha$; observe that $\alpha - \beta \in \mathbb{N}^m$. Then $\mathbf{x}^\alpha = \mathbf{x}^\beta \mathbf{x}^{\alpha-\beta} \in I$. Since N is finite, it follows that all but finitely many monomials are in I.
(ii) Let I be the ideal generated by the set $\{\mathbf{x}^\alpha : \alpha \in T\}$. Suppose $\alpha \in T^*$. Then $\alpha \not\geq \beta$ for all $\beta \in T$. It follows that $\mathbf{x}^\alpha \notin I$. Since $K[\mathbf{x}]/I$ is finite-dimensional, all but finitely many monomials must be in I. It follows that T^* is finite. If $T^* = \emptyset$, then $0 \in T$. Since T is an antichain, $T = \{0\}$. But then $1 \in I$, so T is not full, so T^* must be nonempty. Finally, we know that $\gamma \not\leq \alpha$ for all $\gamma \in T$. If $\beta \in \mathbb{N}^m$ such that $\beta \leq \alpha$, then we must have $\gamma \not\leq \beta$ for all $\gamma \in T$. Therefore $\beta \in T^*$. It follows that T^* is Young-like.
(iii) Suppose $\alpha \in N$. If $\beta \in N^*$ and $\beta \leq \alpha$, then $\beta \in N$. It follows that $\beta \not\leq \alpha$ for all $\beta \in N^*$. Therefore $\alpha \in (N^*)^*$, whence $N \subseteq (N^*)^*$.

Conversely, suppose $\alpha \in (N^*)^*$. Then $\beta \nleq \alpha$ for all $\beta \in N^*$. Since N^* is the set of minimal elements of $\mathbb{N}^m \setminus N$, it follows that $\alpha \in N$, whence $(N^*)^* \subseteq N$.

(iv) Suppose $\gamma \in T$. Suppose $\alpha \in \mathbb{N}^m$ and $\alpha < \gamma$. Since T is an antichain, $\beta \nleq \alpha$ for all $\beta \in T$. Therefore $\alpha \in T^*$. It follows that γ is minimal in $\mathbb{N}^m \setminus T^*$, whence $T \subseteq (T^*)^*$.

Conversely, suppose $\gamma \in (T^*)^*$. Then γ is minimal in $\mathbb{N}^m \setminus T^*$. Note that $\mathbb{N}^m \setminus T^* = \{\alpha \in \mathbb{N}^m : \exists \beta \in T \ \beta \leq \alpha\}$, so there exists $\beta \in T$ such that $\beta \leq \gamma$. Note that $\beta \notin T^*$. If $\gamma \notin T$, then $\beta < \gamma$, contradicting the minimality of γ. It follows that $\gamma \in T$, whence $(T^*)^* \subseteq T$. □

Theorem 4.10. *Fix a Young-like set $N \subseteq \mathbb{N}^m$. Suppose that $B = K[\mathbf{x}]/I$, where I is the ideal generated by $\mathbf{x}^{N^*}$ and $\epsilon = (\mathbf{x}^\alpha + I)_{\alpha \in N}$. Let R be a K-algebra. Then the following are equivalent:*

(i) *(R, ∂) is a $\mathcal{D}$-ring,*
(ii) *∂ is a generalized derivation of type N on R.*

Proof. Let $e\colon R \to \mathcal{D}(R)$ be the map given by

$$e(r) = \sum_{\alpha \in N} \partial_\alpha(r) \otimes \epsilon_\alpha$$

(i) $\implies$ (ii). Since (R, ∂) is a $\mathcal{D}$-ring, e is a K-algebra homomorphism. We must show that the conditions of Definition 4.5 are satisfied.

(a) Immediate from Lemma 2.2.
(b) For any $p, q \in R$,

$$\begin{aligned}\sum_{\alpha \in N} \partial_\alpha(pq) \otimes \epsilon_\alpha &= e(pq) \\ &= e(p)e(q) \\ &= \sum_{\beta,\gamma \in N} \partial_\beta(p)\partial_\gamma(q) \otimes \epsilon_\beta\epsilon_\gamma \\ &= \sum_{\alpha \in N} \sum_{\beta+\gamma=\alpha} \partial_\beta(p)\partial_\gamma(q) \otimes \epsilon_\alpha.\end{aligned}$$

The result follows from matching coefficients.

(c) By the choice of basis ϵ,

$$1_R \otimes \epsilon_0 = 1_R \otimes 1_B = e(1_R) = \sum_{i=0}^{\ell-1} \partial_i(1_R) \otimes \epsilon_i$$

and therefore $\partial_0(1_R) = 1_R$.

(ii) $\Longrightarrow$ (i). We must show that e is a K-algebra homomorphism. By Definition 4.5 and Lemma 4.6, it suffices to show that e is multiplicative. For any $p, q \in R$,

$$\begin{aligned} e(pq) &= \sum_{\alpha \in N} \partial_\alpha(pq) \otimes \epsilon_\alpha \\ &= \sum_{\alpha \in N} \sum_{\beta+\gamma=\alpha} \partial_\beta(p)\partial_\gamma(q) \otimes \epsilon_\alpha \\ &= \sum_{\alpha \in N} \sum_{\beta+\gamma=\alpha} \partial_\beta(p)\partial_\gamma(q) \otimes \epsilon_\beta \epsilon_\gamma \\ &= \sum_{\beta,\gamma \in N} \partial_\beta(p)\partial_\gamma(q) \otimes \epsilon_\beta \epsilon_\gamma \\ &= e(p)e(q). \end{aligned}$$

Therefore (R, ∂) is a $\mathcal{D}$-ring. □

Corollary 4.11. *Suppose that B is isomorphic to a K-algebra B' of type N^* for some Young-like set $N \subseteq \mathbb{N}^m$ and (R, ∂) is a $\mathcal{D}$-ring. Then ∂ is equivalent to a generalized derivation of type N on R.*

Proof. By Proposition 4.2, there exists a matrix $M \in \mathrm{GL}_\ell(K)$ such that $(R, M\partial)$ is a $\mathcal{D}'$-ring. By definition, ∂ is equivalent to $M\partial$. Since B' is of type N^*, every $\mathcal{D}'$-operator is a generalized derivation of type N, and in particular, $M\partial$ is a generalized derivation of type N. □

5. Competency

The notion of competency of a local K-algebra was introduced in [1] as a tool in the classification of $\mathcal{D}$-operators which are not equivalent to generalized derivations. Conjecture 20 of [1] claims that every finite-rank local K-algebra A is isomorphic to a competetent K-algebra when K is algebraically closed. It is easy to see from Table 1 of [5] that the conjecture holds when the rank of A is no more than 6. We do not prove Conjecture 20, but we show that a weaker version always holds, whether or not K is algebraically closed. Additionally, we demonstrate that for a complete classification of $\mathcal{D}$-operators up to equivalence it is sufficient to consider only the case when B is local.

Proposition 5.1. *Suppose that $B \simeq B_1 \oplus_K B_2$ for some K-algebras B_1 and B_2 and that ϵ corresponds to the union of bases ϵ^1 and ϵ^2 for B_1 and B_2 respectively. Let $\mathcal{D}_1 = (B_1, \epsilon^1)$ and $\mathcal{D}_2 = (B_2, \epsilon^2)$. Then the following are equivalent:*

(i) *(R, ∂) is a $\mathcal{D}$-ring,*
(ii) *(R, ∂) is both a $\mathcal{D}_1$-ring and a $\mathcal{D}_2$-ring.*

Proof. Let $\dim_K B_1 = \ell_1$ and $\dim_K B_2 = \ell_2$; note that $\ell_1 + \ell_2 = \ell$. Without loss of generality, assume that $B = B_1 \oplus_K B_2$ and $\epsilon^1 = (\epsilon_0, \ldots, \epsilon_{\ell_1 - 1})$, while $\epsilon^2 = (\epsilon_{\ell_1}, \ldots, \epsilon_{\ell-1})$. Let $\mathcal{D}_1 = (B_1, \epsilon^1)$ and $\mathcal{D}_2 = (B_2, \epsilon^2)$. Observe that

$$\mathcal{D}(R) = R \otimes_K (B_1 \oplus B_2) \simeq (R \otimes_K B_1) \oplus_K (R \otimes_K B_2) = \mathcal{D}_1(R) \oplus_K \mathcal{D}_2(R).$$

Without loss of generality, we also assume that $\mathcal{D}_1(R) \oplus_K \mathcal{D}_2(R) = \mathcal{D}(R)$.

Define $e_1 \colon R \to \mathcal{D}_1(R)$ and $e_2 \colon R \to \mathcal{D}_2(R)$ by

$$e_1(r) = \sum_{i=0}^{\ell_1 - 1} \partial_i(r) \otimes \epsilon_i$$

and

$$e_2(r) = \sum_{i=\ell_1}^{\ell - 1} \partial_i(r) \otimes \epsilon_i.$$

(i) $\Longrightarrow$ (ii) Let $\phi \colon \mathcal{D}(R) \to \mathcal{D}_1(R)$ be the canonical projection map onto the first coordinate via $\mathcal{D}_1(R) \oplus_K \mathcal{D}_2(R)$. Note that ϕ is a K-algebra homomorphism.

We claim that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\;e\;} & \mathcal{D}(R) \\ & \searrow^{e_1} & \downarrow{\scriptstyle \phi} \\ & & \mathcal{D}_1(R) \end{array}$$

It follows immediately from the definition of direct sum that for any $r \in R$,

$$\phi(e(r)) = \phi\left(\sum_{i=0}^{\ell-1} \partial_i(r) \otimes \epsilon_i\right) = \sum_{i=0}^{\ell_1 - 1} \partial_i(r) \otimes \epsilon_i = e_1(r),$$

so e_1 is a K-algebra homomorphism, and (R, ∂) is a $\mathcal{D}_1$-ring. Similarly, (R, ∂) is a $\mathcal{D}_2$-ring.

(ii) $\Longrightarrow$ (i) Since $e = e_1 + e_2$, and e_1 and e_2 are K-algebra homomorphisms, e is linear. By the definition of direct sum, $\epsilon_i \epsilon_j = 0$ whenever $0 \leq i < \ell_1$ and $\ell_1 \leq j < \ell$. Therefore

$$\begin{aligned} e(p)e(q) &= \left(\sum_{i=0}^{\ell-1} \partial_i(p) \otimes \epsilon_i\right)\left(\sum_{j=0}^{\ell-1} \partial_j(q) \otimes \epsilon_j\right) \\ &= \sum_{i=0}^{\ell_1-1} \sum_{j=0}^{\ell_1-1} \partial_i(p)\partial_j(q) \otimes \epsilon_i\epsilon_j + \sum_{i=\ell_1}^{\ell-1} \sum_{j=0}^{\ell-1} \partial_i(p)\partial_j(q) \otimes \epsilon_i\epsilon_j \\ &= e_1(p)e_1(q) + e_2(p)e_2(q) \\ &= e_1(pq) + e_2(pq) \\ &= e(pq). \end{aligned}$$

Therefore e is a K-algebra homomorphism, so (R, ∂) is a $\mathcal{D}$-ring. □

Fact 5.2. Every finite-rank K-algebra is isomorphic to a finite direct sum of local K-algebras.

Corollary 5.3. *Suppose that (R, ∂) is a $\mathcal{D}$-ring. Then there exists a local K-algebra B', a basis ϵ' for B', and a $\mathcal{D}'$-operator ∂' on R such that ∂ is equivalent to ∂'.*

Proof. By Fact 5.2, there exist local K-algebras $B_1, \ldots, B_n$ such that $B \simeq \bigoplus_{i=1}^{n} B_i$. Let M be the change of basis matrix from ϵ to some basis for B corresponding to the union of bases $\epsilon^1, \ldots, \epsilon^n$ for B. By Propositions 4.2 and 5.1, $(R, M\partial)$ is a $\mathcal{D}_1$-ring. Setting $B' = B_1$, $\epsilon' = \epsilon^1$, and $\partial' = M\partial$ proves the result. □

With these results behind us, we are ready to proceed to the discussion of competency. Recall the following definitions (repeated here to correct minor errors) from [1]:

Definition 5.4. Let $N \subseteq \mathbb{N}^m$ be a Young-like set. If $\alpha \in N$ satisfies $\alpha + \beta \notin N$ for all nonzero $\beta \in \mathbb{N}^m$, then we say that α is a *boundary point* of N. Otherwise, we say that α is an *interior point* of N. The set of boundary points of N, called the *boundary* of N, is denoted by ∂N; similarly, the *interior* of N is denoted by $\mathring{N}$.

Definition 5.5. Let I be an ideal of $K[\mathbf{x}]$ such that $K[\mathbf{x}]/I$. Let $A = K[\mathbf{x}]/I$. We say that A is *competent* if there exists a Young-like set $N \subseteq \mathbb{N}^m$ such that

(a) $\mathbf{x}^\alpha \in I$ for all $\alpha \in \mathbb{N}^m \setminus N$,
(b) there exists a set $S \subseteq \partial N$ such that $\{\mathbf{x}^\alpha + I : \alpha \in \partial N\} \subseteq \mathrm{Span}_K\{\mathbf{x}^\alpha + I : \alpha \in S\}$, the elements $(\mathbf{x}^\alpha + I)_{\alpha \in S \cup \mathring{N}}$ are distinct, and the set $\{\mathbf{x}^\alpha + I : \alpha \in S \cup \mathring{N}\}$ is linearly independent in A.

It was noted in Definition 14 of [1] that if A is competent, then $\{\mathbf{x}^\alpha + I : \alpha \in S \cup \mathring{N}\}$ is a basis for A, and $S \cup \mathring{N}$ is Young-like.

Remark 5.6. Proposition 17 of [1] showed that when B is local and competent and $(\partial'_\beta)_{\beta \in N}$ are generalized derivations of type N on some ring R, then defining operators $(\partial_\alpha)_{\alpha \in S \cup \mathring{N}}$ on R by

$$\partial_\alpha = \partial'_\alpha + \sum_{\beta \in \partial N \setminus S} \lambda_\alpha^\beta \partial'_\beta,$$

where $\lambda_\alpha^\beta \in K$ is the unique scalar such that

$$\mathbf{x}^{\alpha+\beta} + I = \sum_{\gamma \in \mathring{N} \cup S} \lambda_\alpha^\beta \mathbf{x}^\gamma + I$$

gives a $\mathcal{D}$-ring (R, ∂).

We now state a weaker condition than competency and show that it holds for all local K-algebras.

Definition 5.7. Define a total order $\preceq$ on $\mathbb{N}^m$ as follows. Let $\alpha, \beta \in \mathbb{N}^m$. Assume that $|\alpha| \le |\beta|$ and $\alpha \ne \beta$.

- If $|\alpha| < |\beta|$, then $\alpha \prec \beta$.
- If $|\alpha| = |\beta|$, let $\gamma = \alpha - \beta$. Let ℓ be minimal such that $\gamma_\ell \ne 0$. If $\gamma_\ell < 0$, then $\alpha \prec \beta$. If $\gamma_\ell > 0$, then $\alpha \succ \beta$.

Note that $\preceq$ in fact defines a well-ordering on $\mathbb{N}^m$.

Notation 5.8. We define the following notation for any $N \subseteq \mathbb{N}^m$, $\beta \in \mathbb{N}^m$, I an ideal of $K[\mathbf{x}]$.

- $\|N\| \coloneqq \sup\{|\alpha| : \alpha \in N\}$.
- For $i \in \mathbb{N}$, $N_i \coloneqq \{\alpha \in N : |\alpha| = i\}$ and $N_{\ge i} \coloneqq \{\alpha \in N : |\alpha| \ge i\}$. $N_{\le i}$, $N_{>i}$, and $N_{<i}$ are similar.
- For $\beta \in \mathbb{N}^m$, $N_{\prec\beta} \coloneqq \{\alpha \in N : \alpha \prec \beta\}$. $N_{\preceq\beta}$, $N_{\succ\beta}$, and $N_{\succeq\beta}$ are similar.
- $\mathbf{x}^N \coloneqq \{\mathbf{x}^\alpha : \alpha \in N\}$ and $\mathbf{x}^N + I \coloneqq \{x^\alpha + I : \alpha \in N\}$.

Note that while $|\mathbf{x}^N| = |N|$, it it not necessarily the case that $|\mathbf{x}^N + I| = |N|$. We shall also denote the m-tuple $(0, 0, \ldots, 0)$ by 0.

Lemma 5.9. *Let $\alpha, \beta \in \mathbb{N}^m$. If $\alpha \prec \beta$, then $\alpha + \eta \prec \beta + \eta$ for every $\eta \in \mathbb{N}^m$.*

Proof. Let $\eta \in \mathbb{N}^m$. If $|\alpha| < |\beta|$, then $|\alpha + \eta| = |\alpha| + |\eta| < |\beta| + |\eta| = |\beta + \eta|$, so $\alpha + \eta \prec \beta + \eta$. If $|\alpha| = |\beta|$, let $\gamma = \alpha - \beta$ and let $\ell \in \mathbb{N}$ be minimal such that $\gamma_\ell \neq 0$. Since $\alpha \prec \beta$, we must have $\gamma_\ell < 0$. Note that $(\alpha + \eta) - (\beta + \eta) = \alpha - \beta = \gamma$ and $|\alpha + \eta| = |\beta + \eta|$, so $\alpha + \eta \prec \beta + \eta$. □

Definition 5.10. Let N be a finite subset of $\mathbb{N}^m$. Suppose that $\mathcal{P}(N)$ holds for some property $\mathcal{P}$. We say that N is *maximal* with respect to $\preceq$ and $\mathcal{P}$ if for all finite subsets M of $\mathbb{N}^m$ such that $\mathcal{P}(M)$ holds, one of the following holds:

(a) $M \subseteq N$,
(b) writing $N \setminus (N \cap M) = \{n_1 \prec \cdots \prec n_a\}$ and $M \setminus (N \cap M) = \{m_1 \prec \cdots \prec m_b\}$, we have $n_i \succeq m_i$ for all $1 \leq i \leq \min(a, b)$.

Remark 5.11. We are particularly interested in constructing monomial bases, which can be indexed by subsets of $\mathbb{N}^m$ corresponding to powers of monomials, for quotients of ideals of polynomial algebras. Given such a K-vector space and a finite set $N \subseteq \mathbb{N}^m$ which corresponds to a spanning set, one can construct a basis maximal with respect to $\preceq$ by iterating through N in descending order according to $\succeq$, adding elements if their corresponding monomials are not in the span of their successors. This fact will be used in the proof of Propositions 5.13 and 5.14.

Definition 5.12. Let $A = K[\mathbf{x}]/I$ be a K-algebra. We say that A is *weakly competent* if there exists a Young-like set M such that

(a) the set $\{\mathbf{x}^\alpha + I : \alpha \in M\}$ is a basis for A
(b) $\mathbf{x}^\alpha \in I$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| > \|M\|$
(c) for all $\beta \in \mathbb{N}^m$, $\mathbf{x}^\beta \in \operatorname{Span}_K \mathbf{x}^{M_{\succeq \beta}}$.

Proposition 5.13. *Let $B = K[\mathbf{x}]/I$ be a K-algebra. If B is competent, then B is weakly competent.*

Proof. Suppose that B is competent. Let N be as in Definition 5.5. Choose $S \subseteq \partial N$ to be maximal under $\preceq$ such that condition (b) of Definition 5.5 holds. Let $M = S \cup \mathring{N}$. We will show that M satisfies the conditions of Definition 5.12.

(a) Immediate from Definition 5.5.
(b) Suppose that $\alpha \in \mathbb{N}^m$ and $|\alpha| > \|M\|$. Consider the case when $\alpha \in N$. Since $\alpha \notin M$, $\alpha \notin \mathring{N}$ and $\alpha \notin S$; hence, $\alpha \in \partial N \setminus S$. If $\mathbf{x}^\alpha \notin I$, then there is some element $\beta \in S$ such that $(M \setminus \{\beta\}) \cup \{\alpha\}$ corresponds to a basis for B. Since $|\alpha| > \|M\|$, this contradicts the maximality of S with respect to $\preceq$. Thus either $\mathbf{x}^\alpha \in I$ or $\alpha \notin N$, and in the latter case we must have $\mathbf{x}^\alpha \in I$ by competency.
(c) The result is trivial for $\beta \in \mathbb{N}^m \setminus N$ and $\beta \in \mathring{N} \cup S$. This leaves only the case when $\beta \in \partial N \setminus S$, which follows by the maximality of S. □

Theorem 5.14. *Every finite-rank, local K-algebra is isomorphic to a weakly competent K-algebra.*

Proof. Let A be a finite-rank, local K-algebra. Let $\mathfrak{m}$ be the maximal ideal of A. Let $m = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$. Let $v_1, v_2, \ldots, v_m \in \mathfrak{m}$ such that $\{v_1 + \mathfrak{m}^2, v_2 + \mathfrak{m}^2, \ldots, v_m + \mathfrak{m}^2\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$.

For each $i \in \mathbb{N}$, define $S_i = \{\mathbf{v}^\alpha : \alpha \in \mathbb{N}^m_i\}$. Note that $S_i \subseteq \mathfrak{m}^i$. Let $\bar{S}_i = \{x + \mathfrak{m}^{i+1} : x \in S_i\}$. We claim that $\bar{S}_i$ spans $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. This is clearly the case for $i = 0, 1$. Inductively, assume that $\bar{S}_i$ spans $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ for some $i \in \mathbb{N}$. Then every element of $\mathfrak{m}^i$ can be written as $\sum_{\alpha \in \mathbb{N}^m_i} b_\alpha \mathbf{v}^\alpha + u$, where each $b_\alpha \in K$ and $u \in \mathfrak{m}^{i+1}$. Also, every element of $\mathfrak{m}$ can be written as $\sum_{j=1}^m c_j v_j + w$, where each $c_j \in K$ and $w \in \mathfrak{m}^2$. Multiplying these sums together shows that every element of $\mathfrak{m}^{i+1}$ can be written as $\sum_{\beta \in \mathbb{N}^m_{i+1}} d_\beta \mathbf{v}^\beta + y$, where each $d_\beta \in K$ and $y \in \mathfrak{m}^{i+2}$. Hence $\bar{S}_i$ spans $\mathfrak{m}^{i+1}/\mathfrak{m}^{i+2}$, and the desired result follows.

Let $M_i \subseteq \mathbb{N}^m_i$ be maximal under $\preceq$ such that $\mathbf{v}^{M_i} + \mathfrak{m}^{i+1}$ forms a basis for $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Let $T_i = \mathbf{v}^{M_i}$ and $\bar{T}_i = \mathbf{v}^{M_i} + \mathfrak{m}^{i+1}$. Let $M = \bigcup_{i \in \mathbb{N}} M_i$. We claim that M is Young-like. We will show that for all $\alpha \in \mathbb{N}^m$, if there exists $\beta < \alpha$ with $\beta \notin M$, then $\alpha \notin N$, which is an equivalent condition. Accordingly, let $\alpha \in \mathbb{N}^m$ and suppose that such a β exists. Then $\beta \notin M_{|\beta|}$, so $\mathbf{v}^\beta + \mathfrak{m}^{|\beta|+1} \in \operatorname{Span}_K \bar{T}_i$. By minimality of M_i, $\mathbf{v}^\beta + \mathfrak{m}^{|\beta|+1}$ is in fact in the span of elements with component degree less than β under $\prec$. Since $\alpha > \beta$, we have $\mathbf{v}^\alpha = \mathbf{v}^\beta \mathbf{v}^\gamma$ for some $\gamma \in \mathbb{N}^m$ with $|\gamma| = |\alpha - \beta|$. By Lemma 5.9, $\mathbf{v}^\alpha + \mathfrak{m}^{|\alpha|+1}$ is in the span of elements with component degree less than α under $\prec$, so $\alpha \notin M_{|\alpha|}$; hence, $\alpha \notin M$. This establishes that M is Young-like.

We now show that $\mathbf{v}^\alpha = 0$ for all $\alpha \in \mathbb{N}^m$ with $\mathbf{v}^\alpha > \|M\|$. Suppose that $\eta \in \mathbb{N}^m$ and $|\eta| > \|M\|$. Then for all $\gamma \in \mathbb{N}^m$ with $|\gamma| = |\eta|$, $\gamma \notin M \supseteq M_{|\eta|}$. By construction, this implies that $M_{|\eta|} = \emptyset$. Therefore $T_{|\eta|} = \mathbf{v}^{M_{|\eta|}} = \emptyset$, so $\bar{M}_{|\eta|} = \emptyset$. Since $\bar{M}_{|\eta|}$ is a basis for $\mathfrak{m}^{|\eta|}/\mathfrak{m}^{|\eta|+1}$, it follows that $\mathfrak{m}^{|\eta|}/\mathfrak{m}^{|\eta|+1} = 0$. Therefore $\mathfrak{m}^{|\eta|} = \mathfrak{m}^{|\eta|+1} = 0$. Hence $\mathbf{v}^\eta = 0$.

We now show that $\mathbf{v}^M$ is a basis for A. Since A is finitely generated as a K-module, A is Artinian; hence, $\mathfrak{m}$ is nilpotent. It follows easily that $\mathbf{v}^M$ spans A. As for linear independence, suppose that $\sum_{\alpha \in N} c_\alpha \mathbf{v}^\alpha = 0$, where each $c_\alpha \in K$. We claim that $c_\alpha = 0$ for all $\alpha \in M$. Reducing this sum modulo $\mathfrak{m} = \mathfrak{m}^1$, we have that $c_0 \bar{1} = 0$ in $\mathfrak{m}^0/\mathfrak{m}^1$; since $\{1 + \mathfrak{m}^1\}$ is a basis for $\mathfrak{m}^0/\mathfrak{m}^1$, we must have $c_0 = 0$. Inductively, assume that there is some $i \in \mathbb{N}_{\geq 1}$ such that $c_\alpha = 0$ for all $\alpha \in \mathbb{N}^m_{<i}$. Then we have $\sum_{\alpha \in N_{\geq i}} c_\alpha \mathbf{v}^\alpha = 0$. Reducing modulo $\mathfrak{m}^{i+1}$, we have $\sum_{\alpha \in N_i} c_\alpha \bar{\mathbf{v}}^\alpha = 0$ in $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Since the image of $\mathbf{v}^{M_i}$ is a basis for $\mathfrak{m}^i/\mathfrak{m}^{i+1}$, we must have $c_\alpha = 0$ for all $\alpha \in M_i$. Therefore $c_\alpha = 0$ for all $\alpha \in M$, which shows that $\mathbf{v}^M$ is linearly independent.

We now show that for all $\alpha \in \mathbb{N}^m$, $\mathbf{v}^\alpha \in \operatorname{Span}_K \mathbf{v}^{M_{\geq |\alpha|}}$. Suppose $\alpha \in \mathbb{N}^m$. If $\mathbf{v}^\alpha = 0$, then the result is trivial. Otherwise, let ℓ be maximal such that $\mathbf{v}^\alpha \in \mathfrak{m}^\ell$. (Since $\mathfrak{m}$ is nilpotent, such a maximal ℓ must exist.) Then $\mathbf{v}^\alpha = \sum_{\gamma \in M_\ell} c_\gamma \mathbf{v}^\gamma + w$ for some scalars c_γ and $w \in \mathfrak{m}^{\ell+1}$, since the image of $\mathbf{v}^{M_\ell}$ is a basis for $\mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}$. It follows easily that $\mathbf{v}^\alpha \in \operatorname{Span}_K\{\mathbf{v}^{M_{\geq \ell}}\}$. Furthermore, $\mathbf{v}^\alpha \in \mathfrak{m}^{|\alpha|}$, so by maximality, $\ell \geq |\alpha|$. Therefore $\mathbf{v}^\alpha \in \operatorname{Span}_K\{\mathbf{v}^{M_{\geq |\alpha|}}\}$.

Define $\phi\colon K[\mathbf{x}] \to A$ by $1 \mapsto 1$ and $x_i \mapsto v_i$ for $1 \leq i \leq m$ and extend by linearity and distributivity. Then ϕ is a surjective K-algebra homomorphism, so $A \simeq K[\mathbf{x}]/I$, where $I := \ker \phi$. By the above construction, $K[\mathbf{x}]/I$ is weakly competent. □

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